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Metrics for Probability Distributions and the Trend to Equilibrium for Solutions of the Boltzmann Equation

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This paper deals with the trend to equilibrium of solutions to the space-homogeneous Boltzmann equation for Maxwellian molecules with angular cutoff as well as with infinite-range forces. The solutions are considered as densities of probability distributions. The Tanaka functional is a metric for the space of probability distributions, which has previously been used in connection with the Boltzmann equation. Our main result is that, if the initial distribution possesses moments of order $2+\varepsilon$, then the convergence to equilibrium in his metric is exponential in time. In the proof, we study the relation between several metrics for spaces of probability distributions, and relate this to the Boltzmann equation, by proving that the Fourier-transformed solutions are at least as regular as the Fourier transform of the initial data. This is also used to prove that even if the initial data only possess a second moment, then $\int_{|v|>R} f(v, t)\, |v|^2\, dv \to 0$ as $R \to \infty$, and this convergence is uniform in time.

KEY WORDS: Boltzmann equation; Fourier transform; probability measures; weak convergence; Prokhorov metric; bivariate distributions with given marginals; Tanaka functional.

1. INTRODUCTION

In this paper we compute bounds on the rate of approach to equilibrium in a metric equivalent to weak*-convergence of measures by solutions to the spatially homogeneous Boltzmann equation for Maxwell molecules and to the noncutoff Kac equation recently introduced by Desvillettes.[13] The results are of interest for the theories both of the spatially dependent Boltzmann equation and the asymptotic equivalence of the latter equation and those of fluid dynamics.

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The rate at with the solution to the Boltzmann equation approaches equilibrium in the strong L^1-sense has been recently investigated by Carlen and Carvalho, first for a model Boltzmann equation[8] and then for a gas of rigid spheres.[9] The main tool was a new quantitative entropy production inequality. With this tool, they developed a method for computing a bound on the time it takes a solution that starts arbitrarily far from equilibrium to reach any given neighborhood of the equilibrium itself. Once an appropriately small neighborhood of the equilibrium is reached, methods based on the spectral theory of the linearized collision operator can be applied, to provide control over the rate of the final approach to equilibrium.[2]

The stability results obtained by Arkeryd[2] are valid only for intermolecular forces harder than Maxwellian ones, which also have an angular cutoff. These results were extended to the Maxwellian case in ref. 38, but also there an angular cutoff is required.

The first asymptotic results on solutions far from equilibrium refer to the spatially homogeneous Maxwellian case. The first one goes back to Ikenberry and Truesdell,[24] who proved that all higher moments that exist initially converge exponentially to the corresponding ones of the equilibrium solution. The second is due to Tanaka,[35] who obtained stability theorems for initial values with equal initial mass in a metric equivalent to weak*-convergence of measures. The question of rate of convergence is not addressed there.

The proofs in ref. 24 and 35 depend on the possibility of particularly exact computations for a gas of Maxwellian molecules, and the same is true for the first of the papers by Carlen and Carvalho.[8]

The basic idea in this paper is to consider the Fourier transform of the Boltzmann equation for Maxwell molecules. The resulting equation, which was introduced by Bobylev,[5,6] is considerably easier to handle because the dimension of the integral in the collision operator is lower. In a recent investigation, Pulvirenti[30] studied this equation, and in particular she found a simple, direct proof of Tanaka's existence and uniqueness result.

In the final part of this introductory section, we will give a more precise statement of the problem, and present the Boltzmann equation and the simple, one-dimensional Kac equation. Moreover, the Tanaka metric will be defined, and after that we can state the main results of the paper.

Then in Section 2 we consider the kinetic equations in greater detail, and in particular the Fourier-transformed versions. Our main interest is the asymptotic behavior of the Fourier transforms of the solutions, and for the Kac equation the most important result is that $\sup_\xi |\hat{f}(\xi, t) - \hat{\omega}(\xi)|/|\xi|^{\alpha+2}$ converges exponentially to 0 as time goes to infinity, if $\hat{\omega}$ is the Fourier transform of the equilibrium solution. This bolds for all $\alpha > 0$ if this quantity is bounded at $t = 0$. A similar result holds for the Boltzmann equation for Maxwell molecules, but there special care has to be taken with the nonsymmetric part of the pressure tensor. The result there is that $\hat{f}(\xi, t) - \hat{\omega}(\xi) = \Phi_1(\xi, t) + \hat{P}(\xi, t)$, where Φ_1 behaves exactly as $\hat{f} - \hat{\omega}$ does for the Kac equation, and where $\hat{P}(\xi, t) = \mathcal{O}(|\xi|^2)$ and decays exponentially with time. For both equations, the results remain valid in the noncutoff case (also for the noncutoff Kac equation, which was introduced by Desvillettes[13]). For the one-dimensional Kac equation, it makes sense to discuss the odd and even parts of the solutions, and as a byproduct of the above discussion we find that the odd part f_0 of a solution of the classical Kac equation satisfies the equation $\partial f_0(v, t)/\partial t + f_0(v, t) = 0$.

The results from Section 2 are slightly generalized in Section 3, where the condition that the initial data possess more than two moments is relaxed. In fact, most of the results from Section 2 hold also in this case, and all that must be added is a discussion concerning the relation between integrability at infinity of a function and the regularity of its Fourier transform. An immediate consequence of this is that the solutions of the Boltzmann equation for Maxwellian molecules are uniformly integrable at infinity, i.e., that, given $\varepsilon > 0$, there is an $R > 0$ such that $\int_{|v| > R} f(v, t)\, dv < \varepsilon$ for all $t \geqslant 0$.

In Sections 4 and 5 the Tanaka metric for probability distributions is introduced, first for the one-dimensional situation, where this metric has a very simple interpretation, and then for the general case. In higher dimensions we find it useful to relate the Tanaka metric to a different metric, the so-called Prokhorov metric. These metrics are discussed in relation with the density function associated with a probability distribution function. For the Kac and Boltzmann equations, this is quite relevant, since if the initial data are densities, then so are the solutions. These results together with the results from Section 2 directly prove the main results, which are stated below.

In the remaining part of this section, we give a presentation of the Boltzmann equation and the Kac equation, and at the very end we are able to state the main results of the paper.

We consider the Boltzmann equation for a monatomic gas and only for space-independent data. In addition we restrict ourselves to the case of Maxwell molecules, so the most general equation treated here is

$$\frac{\partial}{\partial t} f(v, t) = \int_{\mathbf{R}^3} \int_{S^2} [f(v', t) f(w', t) - f(v) f(w)] \times \sigma(n \cdot (v - w)/|v - w|)\, dn\, dw \tag{1.1}$$

together with appropriate initial data f_0. Here appropriate means that the data are positive and satisfy

$$\int_{\mathbf{R}^3} f_0(v)\,dv = 1, \qquad \int_{\mathbf{R}^3} f_0(v)\,|v|^2\,dv = E_0 < \infty$$
$$\int_{\mathbf{R}^3} f_0(v)\log f_0(v)\,dv = H_0 < \infty \tag{1.2}$$

These quantities correspond to the mass, energy, and entropy of the gas, and formal manipulations of (1.1) suggest that the first two are conserved, and that the entropy is nonincreasing. By similar calculations one finds that there is only one more conserved quantity, the momentum,

$$\int_{\mathbf{R}^3} f_0(v)\,v\,dv \tag{1.3}$$

These statements have also been established rigorously, at least for the Maxwell case, which is considered here, and for the so-called cutoff molecules which are described below.

We will also consider other moment conditions on the initial data,

$$\|f_0\|_{1,s} \equiv \int_{\mathbf{R}^3} f_0(v)(1+|v|^2)^{s/2}\,dv < \infty \tag{1.4}$$

as well as a certain bounds on the Fourier transform of f_0 near the origin. This will be described in Section 2.

The right-hand side of (1.1) describes the rate of change of the density function f due to collisions. The probability that a particular collision takes place is given by the rate function σ. This function in general depends also on the relative velocity $|v-w|$, but for Maxwell molecules there is only an angular dependence as in (1.1).

The conservation laws are derived from the fact that the collisions are assumed to be elastic, and therefore to conserve momentum ($v+w=v'+w'$) and energy ($|v|^2+|w|^2=|v'|^2+|w'|^2$). Here, as well as in (1.1), v' and w' denote the velocities of two particles which had the velocities v and w before they collided. The new velocities also depend on the impact parameter n, and they are given by (which is one of the two natural representations)

$$v' = \tfrac{1}{2}(v+w+|v-w|\,n)$$
$$w' = \tfrac{1}{2}(v+w-|v-w|\,n) \tag{1.5}$$

The term "cutoff molecule" is used to signify molecules such that

$$\int_{S^2} \sigma[u \cdot (v-w)/|v-w|]\, du \equiv \bar{\sigma} < \infty$$

When this holds, the Boltzmann equation can be written

$$\frac{\partial}{\partial t} f(v, t) + \bar{\sigma} f(v, t) = \int_{\mathbf{R}^3} \int_{S^2} f(v', t)\, f(w', t)\, \sigma[n \cdot (v-w)/|v-w|]\, dn\, dw$$

and the right-hand side, the gain term, will usually be denoted $Q^+(f, f)(v)$. Much is known about this equation, apart from the existence of a unique solution, for which the conservation laws can be established rigorously. The only equilibrium solutions are the so-called Maxwellians, which are functions of the form $\omega(v) = (2\pi)^{-3/2} \exp(-|v|^2/2)$. A Maxwellian with these coefficients has mass $\|\omega\|_{L^1} = 1$, energy $E(\omega) = 3$, the momentum is zero, and it is the unique limit as t tends to infinity of any solution of the Boltzmann equation if the initial data have the same mass, momentum, and energy.

A recent result which is relevant when studying the entropy production[9] is the construction of a time-independent (for $t \geqslant t_0 > 0$) lower bound of the type $c \exp(-|v|^{2+\varepsilon}/b)$.[31]

In the noncutoff case the situation is more complicated, since then the form (1.1) must be kept, and one is limited to study weak solutions, for which (1.1) makes sense first after multiplication with certain test functions and integration.[1, 16] Apart from the uniqueness and existence results mentioned above, perhaps the result which is most relevant in the present context is the recent proof by Arkeryd[3] that the weak solutions converge strongly to a Maxwellian (which by uniqueness in the case of Maxwell molecules must be the one Maxwellian with the correct mass and energy). New, much simplified proofs of the existence and uniqueness of solutions in the Maxwell case have been obtained recently by Pulvirenti and Toscani.[24] The uniqueness result implies that one is free to construct the weak solutions according to convenience. Here we consider the weak limits of the solutions of (1.6) where the rate function σ has been replaced by

$$\sigma_k(x) = \min(k, \sigma(x)), \qquad \bar{\sigma}_k = \int_{S^2} \sigma_k[n \cdot (v-w)/|v-w|]\, dn \qquad (1.6)$$

What is important to note is that, in order to transfer convergence results from the cutoff case to the noncutoff case, it is necessary to determine how these depend on the exact form of σ.

In spite of the fact that a lot is known about the space-homogeneous Boltzmann equation and its solutions, there are still unsolved problems. For this reason it is interesting to study models of the full equation which are simpler in some ways, but still keep as many of the properties as possible. One such equation is the one-dimensional Kac equation, which was first discussed in ref. 25 and subsequently thoroughly investigated by McKean.[26] The original form is

$$\frac{\partial}{\partial t} f(v, t) + f(v, t) = \int_{-\pi}^{\pi} f(v', t) f(w', t) \frac{1}{2\pi} d\theta \, dw \tag{1.7}$$

where in this case the postcollisional velocities are given by a rotation in the (v, w) plane

$$v' = v \cos\theta - w \sin\theta$$

$$w' = v \sin\theta + w \cos\theta$$

Clearly the structure of this equation is similar to the Boltzmann equation, and also here mass and energy are conserved and the entropy is nonincreasing. However, momentum is not conserved unless it is zero initially, and therefore that will be the only case considered. Also here the only equilibrium solutions are Maxwellians, which in this case should be normalized as $\omega(v) = (2\pi)^{-1/2} \exp(-|v|^2/2)$.

The right-hand side of (1.7), which also for the Kac model will be denoted Q^+, signifies that all collisions are equally probable, and for the Boltzmann equation that would correspond to σ being a constant. Recently Desvillettes introduced a generalized Kac model in which the factor $1/2\pi$ is replaced by a function $\sigma(\theta)$. His goal was to obtain an equation analogous to the noncutoff Boltzmann equation, and therefore he made the assumption that σ has a nonintegrable singularity at $\theta = 0$.

The solutions of (1.1) and of (1.7) are densities, and there are associated probability distributions on $\mathbf{R}^3$ and $\mathbf{R}$, respectively. Our main results about the asymptotic behavior of the solutions are expressed in terms of a metric on the space of probability distributions, the Tanaka metric. This metric, which is thoroughly discussed in Section 4 for the one-dimensional case and in Section 5 for the higher dimensional case, is defined by

$$T(F, G) = \inf E(|X - Y|^2)$$

Here F and G are two distribution functions on $\mathbf{R}^d$, and the infimum should be taken over all pairs of random variables X and Y taking values in $\mathbf{R}^d$ distributed according to F and G, respectively. Tanaka[34, 35] proved that if F is the distribution function corresponding to a solution f of the Boltzmann or Kac equation, and if Ω is the distribution function corresponding to an equilibrium function, then $T(F(\cdot, t), \Omega) \to 0$ as $t \to \infty$, and this in turn corresponds to weak*-convergence. We prove that if (1.4) holds for some $s > 2$, then

$$T(F(\cdot, t), \Omega) \leqslant C_1 e^{-C_2 t} \tag{1.8}$$

The constants C_1 and C_2 can be estimated uniformly in terms of the moment conditions of f_0. Both for the Kac and Boltzmann equations, the same results hold in the cutoff case as in the noncutoff case.

2. THE FOURIER TRANSFORM OF KAC AND BOLTZMANN EQUATIONS

This section deals with the Fourier transform of equations (1.1) and (1.7). The idea of discussing the Fourier transform of these equations is not new. For the Boltzmann equation, Bobylev[5] first made a thorough investigation of the matter. He found that the equation for $\hat{f}$ in certain respects is much simpler than the equation for f, and in particular that the collision operator is less complicated. For a complete review, including references for the pertinent literature on the subject, see Bobylev.[6]

The Kac equation also is simplified by considering the Fourier transform. This simplification has been exploited among others by Gabetta and Toscani[18,21] and more recently by Desvillettes[13] in this proof that the solutions of the noncutoff equation are smooth, and by Gabetta and Pareschi.[19]

Recently, the Fourier transform also proved useful in connection with the numerical approximation of the collision operator.[12,20]

In most of the discussion in this section, we will assume that the rate function σ is normalized, i.e., that $\bar{\sigma} = 1$. This can always be achieved by a change of variables, $t \to \bar{\sigma} t$, and in the cutoff case it is only a matter of convenience to do so. In the construction of weak solutions to the noncutoff equations, however, we actually consider a sequence of equations where the $\bar{\sigma}_k$ is a sequence increasing to infinity. If f_k denote the solutions of the corresponding *normalized* equations, then the weak solution of the noncutoff equation is the weak limit of $f_k(v, \bar{\sigma}_k t)$. The number $\bar{\sigma}$ is a measure of the collision frequency, and when the truncation is lifted in (1.6), this frequency goes to infinity. That in turn is equivalent to a stretching of time in the corresponding normalized equation. But the stretching of time takes place in a sequence of different equations, and our main result concerning the noncutoff equation must be thought of as a balance between the increasing number of collisions as $k \to \infty$ and the fact that each collision is likely to change f less when k is large.

First we consider the simpler Kac equation and its solutions $f(v, t)$. The Fourier transform of f is

$$\hat{f}(\xi, t) = \int_{-\infty}^{\infty} f(v, t)\, e^{-i\xi v}\, dv$$

and the conservation of mass and energy for f translates to

$$\begin{aligned} &\hat{f}(0, t) = 1, && \|\hat{f}(\cdot, t)\|_\infty \leqslant 1 \\ &\hat{f}''_{\xi\xi}(0, t) = -E_0, && \|\hat{f}''_{\xi\xi}(\cdot, t)\|_\infty \leqslant E_0 \end{aligned} \tag{2.1}$$

each one being valid for all $t \geqslant 0$. The equation which gives the evolution of $\hat{f}$ can be found by multiplying (1.7) with $e^{-i\xi v}$ and integrating. This gives

$$\frac{\partial}{\partial t}\hat{f}(\xi, t) + \hat{f}(\xi, t) = \int_{-\pi}^{\pi} \hat{f}(\xi \cos\theta)\, \hat{f}(\xi \sin\theta)\, \sigma(\theta)\, d\theta \tag{2.2}$$

for the general case with a nonconstant rate function, and with σ normalized so that $\int_{-\pi}^{\pi} \sigma(\theta)\, d\theta = 1$. If one makes the assumption that σ is an even function, it is easy to split (2.2) into a system of equations for the real and imaginary parts of $\hat{f}$. Thus we write $\hat{f}(\xi, t) = \phi(\xi, t) + i\psi(\xi, t)$, and obtain for the real part

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(\xi, t) + \phi(\xi, t) = \int_{-\pi}^{\pi} \phi(\xi \cos\theta)\, \phi(\xi \sin\theta)\, \sigma(\theta)\, d\theta \\ &\phi(\xi, 0) = \operatorname{Re} \hat{f}(\xi, 0) \end{aligned} \tag{2.3}$$

and for the imaginary part

$$\begin{aligned} &\frac{\partial}{\partial t}\psi(\xi, t) + \psi(\xi, t) = \int_{-\pi}^{\pi} \psi(\xi \cos\theta)\, \phi(\xi \sin\theta)\, \sigma(\theta)\, d\theta \\ &\psi(\xi, 0) = \operatorname{Im} \hat{f}(\xi, 0) \end{aligned} \tag{2.4}$$

One interesting aspect of this splitting is that the equation for the real part (corresponding of course to the even part of f) satisfies the same equation as $\hat{f}$, and that the equation for ψ is linear, once (2.3) has been solved. In fact, for any σ which has the additional symmetry $\sigma(\theta) = \sigma(\pi - \theta)$, the right-hand side of (2.4) vanishes, which means that $\psi(\xi, t) = \psi_0(\xi)\, e^{-t}$, and the same then holds for the odd part of the solution f.

Inspired by this observation, one would like to obtain a similar result for the real part, and for the imginary part for more general rate functions.

The main estimate for the Kac equation is the following.

Lemma 2.1. Let $\hat{f}(\xi, t)$ be the solution of Eq. (1.7), and assume that the initial data $\hat{f}_0$ satisfy the natural bounds given by (2.1), and assume in addition that $\hat{f}_0(\xi) - \hat{\omega}(\xi) = \mathcal{O}(|\xi|^{2+\alpha})$ as $\xi \to 0$, where $\alpha > 0$ is any given constant. There is a constant $A < 1$ depending on α and on the rate function σ such that

$$\frac{|\hat{f}(\xi, t) - \hat{\omega}(\xi)|}{|\xi|^{2+\alpha}} \leqslant \left\| \frac{\hat{f}(\cdot, 0) - \hat{\omega}}{|\cdot|^{2+\alpha}} \right\|_{\infty} e^{-(1-A)t} \tag{2.5}$$

Remark 2.1. The condition that $\hat{f}_0(\xi) - \hat{\omega}(\xi) = \mathcal{O}(|\xi|^{2+\alpha})$ is satisfied if $\|f\|_{1, 2+\alpha} < \infty$. The boundedness for all times of moments that exist initially was first established for Maxwellian molecules in ref. 24 and in a more general case, including soft and hard potentials, in ref. 16. But even with no extra condition on the initial data (apart from bounded mass and energy), $(\hat{f}(\xi) - \omega(\xi)) = o(|\xi|^2)$. An analogy of Lemma 2.1 for this case will be discussed in Section 3.

Proof of Lemma 2.1. The main tool is the following estimate of the Gronwall type. Assume that g satisfies, for some positive constant $A < 1$,

$$\left| \frac{\partial}{\partial t} g(\xi, t) + g(\xi, t) \right| \leqslant A \, \|g(\cdot, t)\|_{\infty}, \quad \text{a.e. } \xi \tag{2.6}$$

Then

$$\|g(\cdot, t)\|_{\infty} \leqslant \|g(\cdot, 0)\|_{\infty} \, e^{-(1-A)t} \tag{2.7}$$

In order to use this estimate, we define $\Phi(\xi, t) = [\hat{f}(\xi, t) - \omega(\xi)] \, |\xi|^{-(2+\alpha)}$. Then Φ satisfies

$$\frac{\partial}{\partial t} \Phi(\xi, t) + \Phi(\xi, t) = \int_{-\pi}^{\pi} (\Phi(\xi \cos\theta, t) \, |\cos\theta|^{2+\alpha} \hat{f}(\xi \sin\theta, t) + \hat{\omega}(\xi \cos\theta) \, |\sin\theta|^{2+\alpha} \, \Phi(\xi \sin\theta, t)) \, \sigma(\theta) \, d\theta \tag{2.8}$$

With an analogous definition, the initial data Φ_0 are bounded. Then the hypotheses for $\hat{f}_0$ imply that $\|\Phi_0\|_{\infty} < \infty$. But due to (2.1), which hold also for $\hat{\omega}$, and for all $t \geqslant 0$, the right-hand side of (2.8) can be estimated by $A \, \|\Phi(\cdot, t)\|_{\infty}$, where

$$A = \int_{-\pi}^{\pi} (|\cos\theta|^{2+\alpha} + |\sin\theta|^{2+\alpha}) \, \sigma(\theta) \, d\theta \tag{2.9}$$

Since $|\cos\theta|^{2+\alpha}+|\sin\theta|^{2+\alpha}<1$ for $\cos\theta\neq 0, 1$, the integral is strictly smaller than one, and thus the conditions for the Gronwall estimate (2.6), (2.7) are satisfied. ∎

We proceed now to the Boltzmann equation. The Fourier transform here is

$$\hat{f}(\xi, t)=\int_{\mathbf{R}^3} f(v, t)\, e^{-i\xi\cdot v}\, dv$$

where $\xi\cdot v$ denotes the usual inner product. In this case, the conserved quantities are

$$\begin{aligned} \hat{f}(0, t)&=1, & \|\hat{f}(\cdot, t)\|_\infty&\leqslant 1\\ \nabla^2\hat{f}(0, t)&=-E_0, & \|\nabla^2\hat{f}(\cdot, t)\|_\infty&\leqslant E_0 \end{aligned} \tag{2.10}$$

As was the case with the Kac equation, one may now take the Fourier transform of the Boltzmann equation (1.1)[6] to obtain

$$\frac{\partial}{\partial t}\hat{f}(\xi, t)+\hat{f}(\xi, t)=\int_{S^2}\hat{f}(\xi^+, t)\,\hat{f}(\xi^-, t)\,\sigma\left(\frac{n\cdot\xi}{|\xi|}\right) dn\equiv\hat{Q}^+(\hat{f},\hat{f})(\xi, t) \tag{2.11}$$

Here $\xi^\pm=(\xi\pm|\xi|\, n)/2$, and thus ξ^+ and ξ^- lie on a sphere with one of its poles at the origin and the other one at ξ. From a numerical point of view, this means a great reduction in computational effort, since the dimension of integration is reduced from five to two. Another important aspect is that $\hat{Q}^+$ is localized in the sense that $\hat{Q}^+(\hat{f}, \hat{g})(\xi)$ only depends on $\hat{f}(\xi^+)$ and $\hat{g}(\xi^-)$ for $|\xi^\pm|\leqslant|\xi|$.

Our next goal is to obtain an estimate like the one in Lemma 2.1. Actually that is easy: Lemma 2.1 holds exactly as it stands if $\hat{f}$ is interpreted as for the Boltzmann equation. The proof is just like the proof in the one-dimensional case. However, in order to achieve that the initial data satisfy $(\hat{f}_0(\xi)-\hat{\omega}(\xi))=\mathcal{O}(|\xi|^{2+\alpha})$ one must make the assumption that

$$\hat{f}_{\xi_1\xi_1}(0, t)=\hat{f}_{\xi_2\xi_2}(0, t)=\hat{f}_{\xi_3\xi_3}(0, t)$$

and that all mixed derivatives vanish at the origin. This is much more than requiring that f_0 has a given mass, momentum, and energy. Thus the first task will be to investigate the evolution of $\hat{f}_{\xi_i\xi_j}(0, t)$ for $i\neq j$ and of

$\hat{f}_{\xi_i \xi_j}(0, t) - \nabla^2 \hat{f}(0, t)$. This corresponds exactly to studying the nonisotropic part of the pressure tensor,

$$p_{i,j}(t) = \int_{\mathbf{R}^3} f(v, t)(v_i v_j - \delta_{i,j} |v|^2/3)\, dv, \qquad \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

Such a study was carried out already in ref. 24 for real Maxwellian molecules, and the slightly more general situation here can be treated in a similar way. For this computation, it seems that the original form of the equation is most practical. Thus we multiply (1.1) by $v_i v_j$ and integrate to get

$$\frac{\partial}{\partial t} \int_{\mathbf{R}^3} f(v, t)\, v_i v_j\, dv = \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} f(v) f(w)(v_i' v_j' - v_i v_j)\, \sigma\left(\frac{n \cdot q}{|q|}\right) dn\, dv\, dw$$

where q denotes $v - w$, and where the gain term has been rewritten with the change of variables $(v, w) \to (v', w')$ in the usual way (all such details can be found, e.g., in ref. 11). Expanding the expressions for v' and w' [see Eq. (1.5)] gives

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbf{R}^3} f(v, t)\, v_i v_j\, dv &= \frac{1}{4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} f(v) f(w)(w_i w_j - 3 v_i v_j)\, \sigma\left(\frac{n \cdot q}{|q|}\right) dn\, dv\, dw \\ &+ \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} f(v) f(w)(v_i w_j + |q|\, v_i n_j + |q|\, w_i n_j)\, \sigma\left(\frac{n \cdot q}{|q|}\right) dn\, dv\, dw \\ &+ \frac{1}{4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} f(v) f(w)\, |q|^2 n_i n_j\, \sigma\left(\frac{n \cdot q}{|q|}\right) dn\, dv\, dw \end{aligned} \tag{2.13}$$

As before, we assume that $\int_{S^2} \sigma(n \cdot q/|q|)\, dn = 1$, and moreover that σ is an even function. In addition we assume that the mass $\|f\|_{L^1} = 1$, and that the momentum is zero. Then the first term in (2.13) becomes $-\frac{1}{2} \int_{\mathbf{R}^3} f(v)\, v_i v_j\, dv$, and the second term vanishes. The third term can be written

$$\tfrac{1}{4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} f(v) f(w)\, |q|^2 A_{i,j}(q)\, dv\, dw \tag{2.14}$$

where

$$A_{i,j}(q)=\int_{S_2} n_i n_j \sigma\left(\frac{n\cdot q}{|q|}\right) dn\,dv\,dw=\frac{3A'-1}{2\,|q|^2}\,q_i q_j+\frac{1-A'}{2}\,\delta_{i,j} \quad (2.15)$$

The constant A' is given by

$$A'=\int_{S^2} \sigma(u\cdot q/|q|)(u\cdot q/|q|)^2\,dn=2\pi\int_0^\pi \sigma(\cos\theta)\cos^2\theta\sin\theta\,d\theta \quad (2.16)$$

The last expression of (2.15), and the expression for A', which shows that A' really is a constant and does not depend the direction of q, can be obtained by some algebraic manipulations, after expressing the first integral in polar coordinates around $q/|q|$. The normalization of σ means that the integral in (2.16), *without* the factor $\cos^2\theta$, is one, and therefore A' is strictly less than one. Inserting this into (2.13) and expanding $q=v-w$ gives

$$\frac{\partial}{\partial t}\int_{\mathbf{R}^3} f(v,t)\,v_i v_j\,dv=\frac{3(A'-1)}{4}\int_{\mathbf{R}^3} f(v,t)\,v_i v_j\,dv+\delta_{i,j}\frac{1-A'}{4}\int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv$$

which has the solution

$$p_{i,j}(t)=e^{-A_1 t}p_{i,j}(0) \quad (2.17)$$

where $A_1=3(1-A')/4$. Note that these are exact computations, which depend only on the *a priori* knowledge that the mass, momentum, and energy are conserved. We recall again that this result can be found in ref. 24.

On the Fourier transform side, this implies that

$$\hat{f}_{\xi_i\xi_j}(\xi,t)+\tfrac{1}{3}\delta_{i,j}\nabla^2\hat{f}(\xi,t)$$

is bounded (this follows from the conservation of mass and energy), and decays exponentially with the rate $\exp(-A_1 t)$ at the point $\xi=0$.

We can now return to finding an estimate analogous to (2.5) for the Boltzmann equation. As in the proof of Lemma 2.1, let $\Phi(\xi,t)=\hat{f}(\xi,t)-\hat{\omega}(\xi)$. Then Φ satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}\Phi+\Phi=\hat{Q}^+(\Phi,\hat{f})+\hat{Q}^+(\hat{\omega},\Phi)\\&\Phi(\xi,0)=\hat{f}_0(\xi)-\hat{\omega}(0)\end{aligned} \quad (2.18)$$

and the conservation of mass, momentum, and energy imply that

$$\Phi(\xi, t) = -\sum_{i,j} p_{i,j}(t)\, \xi_i \xi_j + o(|\xi|^2) \qquad \text{as} \quad \xi \to 0$$

where the $p_{i,j}$ are given by (2.17) (cf. Remark 2.1). Define $X_{i,j}(\xi) = \xi_i \xi_j$ for $|\xi| \leqslant 1$, and $X_{i,j}(\xi) = 0$ otherwise, and let

$$\hat{P}(\xi, t) = -e^{-A_1 t} \sum_{i,j} p_{i,j}(0)\, X_{i,j}(\xi), \qquad \Phi_1(\xi, t) = \Phi(\xi, t) - \hat{P}(\xi, t) \quad (2.19)$$

Just as in the proof of Lemma 2.1, we assume that $\Phi_1(\xi, t) = \mathcal{O}(|\xi|^{2+\alpha})$ for some $\alpha > 0$ (by making an extra assumption on the moments for f_0). Inserting (2.19) into (2.18) gives an equation for Φ_1,

$$\begin{aligned} \frac{\partial}{\partial t} \Phi_1 + \Phi_1 &= \hat{Q}^+(\Phi_1, \hat{f}) + \hat{Q}^+(\omega, \Phi_1) + \Phi(\hat{P}, \hat{f}) + \hat{Q}^+(\omega, \hat{P}) - \frac{\partial}{\partial t} \hat{P} - \hat{P} \\ &= \hat{Q}^+(\Phi_1, \hat{f}) + \hat{Q}^+(\omega, \Phi_1) \qquad (2.20) \\ &\quad + e^{-A_1 t} \sum_{i,j} p_{i,j}(0) [(1 - A_1)\, X_{i,j} - \hat{Q}^+(X_{i,j}, \hat{f}) + \hat{Q}^+(\omega, X_{i,j})] \\ \Phi_1(\xi, 0) &= \hat{f}_0(\xi) - \hat{\omega}(\xi) - \hat{P}(\xi, 0) = \mathcal{O}(|\xi|^{2+\alpha}) \end{aligned}$$

Using the fact that $\hat{f}(\xi, t) = 1 + \mathcal{O}(|\xi|^2)$ (this holds uniformly in time), and $\hat{\omega} = 1 + \mathcal{O}(|\xi|^2)$, one obtains, by calculations like the ones in (2.13),

$$\begin{aligned} &\hat{Q}^+(X_{i,j}, \hat{f})(\xi) + \hat{Q}^+(\omega, X_{i,j})(\xi) \\ &= \int_{S^2} (\xi_i^+ \xi_j^+ + \xi_i^- \xi_j^-)\, \sigma(n \cdot \xi/|\xi|)\, dn + \mathcal{O}(|\xi|^4) \\ &= \tfrac{1}{2} \xi_i \xi_j + \tfrac{1}{2} |\xi|^2 \int_{S_2} n_i n_j \sigma(n \cdot \xi/|\xi|)\, dn + \mathcal{O}(|\xi|^4) \\ &= \tfrac{1}{2} \xi_i \xi_j + \tfrac{1}{4}(3A' - 1)\, \xi_i \xi_j + \tfrac{1}{4}(1 - A')\, \sigma_{i,j} |\xi|^2 + \mathcal{O}(|\xi|^4) \end{aligned}$$

One can then check that the $\mathcal{O}(|\xi|^2)$ terms in the sum in (2.20) cancel, and consequently that

$$\frac{\partial}{\partial t} \Phi_1(\xi, t) + \Phi_1(\xi, t) = \hat{Q}^+(\Phi_1, \hat{f})(\xi) + \hat{Q}^+(\omega, \Phi_1)(\xi) - e^{-A_1 t} R(\xi, t)$$

where $R(\xi, t)$ and $R(\xi, t)\,|\xi|^{-4}$ are bounded, uniformly in time. Following the procedure from the case of the Kac equation, we define $\Phi_2(\xi, t) = \Phi_1(\xi, t)\,|\xi|^{-(2+\alpha)}$. Then Φ_2 satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\Phi_2(\xi, t) + \Phi_2(\xi, t)\\
&\qquad = \int_{S^2} [\Phi_2(\xi^+, t)\,\hat{f}(\xi^-, t)\,|\cos(\theta/2)|^{2+\alpha}\\
&\qquad\quad + \hat{\omega}(\xi^+)\,\Phi_2(\xi^-, t)\,|\sin(\theta/2)|^{2+\alpha}]\,\sigma(\cos(\theta))\,dn\\
&\qquad\quad + e^{-A_1 t} R(\xi, t)\,|\xi|^{-(2+\alpha)}
\end{aligned}$$

where $\cos(\theta) = n \cdot \xi/|\xi|$. Then

$$\left|\frac{\partial}{\partial t}\Phi_2(\xi, t) + \Phi_2(\xi, t)\right| \leqslant \|\Phi_2(\cdot, t)\|_\infty A'' + C_1 e^{-A_1 t} \tag{2.21}$$

where

$$\begin{aligned}
A'' &= \int_{S^2} [\,|\cos(\theta/2)|^{2+\alpha} + |\sin(\theta/2)|^{2+\alpha}]\,\sigma(\cos\theta)\,dn\\
&= 2\pi \int_{-\pi}^{\pi} [\,|\cos(\theta/2)|^{2+\alpha} + |\sin(\theta/2)|^{2+\alpha}]\,\sigma(\cos\theta)\sin(\theta)\,d\theta
\end{aligned} \tag{2.22}$$

and $C_1 = \sup_{\xi, t} R(\xi, t)\,|\xi|^{2-\alpha}$. Note that $C_1 = 0$ if the pressure tensor of the initial density is isotropic. The constant A'' is smaller than one for the same reason as for A or A'. Then, using an estimate similar to (2.7), (2.5), we may deduce the following lemma.

Lemma 2.2. Let $\hat{f}(\xi, t)$ be the solution of Eq. (1.1), and assume that the initial data $\hat{f}_0$ satisfy the natural bounds given by (2.1). Assume in addition that $\hat{f}_0(\xi) - \hat{\omega}(\xi) - \hat{P}(\xi, 0) = \mathcal{O}(|\xi|^{2+\alpha})$, where $\hat{P}$ is defined by (2.19), and where α is a given positive constant. Then

$$\hat{f}(\xi, t) = \hat{\omega}(\xi) + \hat{P}(\xi, t) + \Phi_1(\xi, t)$$

where

$$|\Phi_1(\xi, t)| \cdot |\xi|^{-(2+\alpha)} \leqslant [C_1 + \|\Phi_1(\cdot, 0)| \cdot |^{-(2+\alpha)}\|_\infty]\,e^{-C_\alpha t}$$

The constant C_1 is the constant from (2.21) [$C_1 = 0$ if $\hat{P}(\xi, 0) = 0$], and $C_\alpha = \min(A_1, (1 - A''))$.

One problem remains to be addressed. Recall that in the noncutoff case, the rate function σ has a nonintegrable singularity at $\cos\theta = 1$, and that the solutions are constructed as the weak limits of the sequence $f_k(v, \bar{\sigma}_k t)$, where $f_k(v, t)$ is the solution of (1.1) [or (1.7)], with σ replaced by $\sigma_k/\bar{\sigma}_k$, which are defined by (1.6). Of course the same substitutions are made in the Fourier-transformed equations (2.2) and (2.11). We must now determine how the constants A, A'', and A_1 depend on k. The following result holds only if the singularity of σ is not too strong, but the conditions given in Lemma 2.3 are exactly the ones that one expect from a physical point of view.

Lemma 2.3. 1. Let $\sigma(\theta)$ be the rate function for the Kac equation, and assume that $|\theta|^2\sigma(\theta) \in L^1([-\pi, \pi])$. Let A_k be defined by (2.9), with σ replaced by σ_k. Then

$$A_k = 1 - \bar{A}_k/\bar{\sigma}_k \tag{2.23}$$

where $0 < \bar{A}_k \to \bar{A} < 1$, for some $\bar{A}$, as $k \to \infty$. The constant $\bar{A}$ depends on σ and on α.

2. The same type of estimates hold for A' and A'' if the rate function for the Boltzmann equation satisfies $|\theta|^2 \sigma(\cos\theta) \in L^1([0, \pi], \sin\theta \, d\theta)$.

Proof. The proof is exactly the same in all cases. From Eq. (2.9),

$$A_k = \frac{1}{\bar{\sigma}_k} \int_{-\pi}^{\pi} \sigma_k(\theta) \, d\theta - \frac{1}{\bar{\sigma}_k} \int_{-\pi}^{\pi} (1 - |\cos\theta|^{2+\alpha} - |\sin\theta|^{2+\alpha}) \, \sigma_k(\theta) \, d\theta$$

$$= 1 - \frac{1}{\bar{\sigma}_k} \int_{-\pi}^{\pi} (1 - |\cos\theta|^{2+\alpha} - |\sin\theta|^{2+\alpha}) \min(\sigma(\theta), k) \, d\theta$$

and the expression multiplying σ_k in the last member is $\mathcal{O}(\theta^2)$, the integral in the last member converges as $k \to \infty$, to some $\bar{A} > 0$. ∎

3. DATA WITH NO HIGHER MOMENTS BOUNDED

In this section we remove the condition that the initial data possess some higher moment, and assume only that the mass and energy are bounded. Let f be any function with bounded mass and energy, i.e., such that $\int_{\mathbf{R}^3} f(v)(1 + |v|^2) \, dv < \infty$. Then there is a strictly increasing function $\phi: \mathbf{R}^+ \to \mathbf{R}^+$, $\phi(r) \to \infty$ as $r \to \infty$, but such that still $f(v) \, \phi(|v|^2) \in L^1(\mathbf{R}^3)$. This means, very loosely speaking, not only that the energy is bounded, but that one has some extra integrability of f for large velocities. Exactly how much must depend on f, and therefore ϕ depends on f. This classical result from real analysis has recently been used in connection with existence proofs for the BGK model of the Boltzmann equation.[32]

The integrability and behavior at infinity of a function f are related to the regularity of the Fourier transform $\hat{f}$, and from the function ϕ one can directly compute a modulus of continuity for the second derivatives of $\hat{f}$. It will be an easy generalization of the results from Section 2 to prove that the solutions of the Boltzmann equation preserve the modulus of continuity for the initial data. The most interesting implication of that is that this implies a uniform integrability at infinity for the solutions: If $f_0(v)(1+|v|^2)\,\phi(|v|)\in L^1(\mathbf{R}^3)$, then $f(v,t)(1+|v|^2)[\phi(|v|)]^{1/2}\in L^1(\mathbf{R}^3)$ for all $t>0$. In fact, we were motivated to study the problem of this section by E. A. Carlen, who has proven that this holds for the Kac equation, and he suggested that the results analogous to those in the previous section could be useful for proving such a result[7] (the result for the Kac model follows from ref. 10, Theorem 2.1, and the remarks thereafter). The result has a direct application in terms of entropy production bounds. In refs. 8 and 9 it is proven that the entropy production rate $dH(f)/dt$ can be estimated in terms of the relative entropy $H(f\mid\omega)$ as $|dH(f)/dt|\geqslant\Psi(H(f\mid\omega))$, where Ψ is a strictly increasing function and $\Psi(0)=0$. This proves that the entropy of solutions $f(t)$ to the Boltzmann equation increases toward its maximum, but only if the same function Ψ can be used for all $f(t)$, $t>0$. Since the construction of Ψ depends on the decay of $\int_{|v|>R}f(v)\,dv$ as $R\to\infty$, the uniform estimates obtained in this section are precisely what is needed to prove that the entropy increases toward its maximum if all initial data with bounded entropy and energy. Using Elmroth's result that all moments of order higher than two remain bounded requires that the initial data possess at least $2+\varepsilon$ moments.

Let us first study the relation between the integrability of a function f and the regularity of $\hat{f}$. The easy part is to estimate the modulus of continuity of the Fourier transform:

Lemma 3.1. Let $0<\phi$: $\mathbf{R}^+\to\mathbf{R}^+$ be a strictly increasing function, and such that $\phi(r)/r$ is decreasing. Let $\psi(\zeta)=1/\phi(2/\zeta)$. Then

$$\int_{\mathbf{R}^3}|f(v)|\,(1+|v|^m)\,\phi(|v|))\,dv<\infty\Rightarrow|D^m\hat{f}(\xi)-D^m\hat{f}(\eta)|\leqslant\psi(|\xi-\eta|) \quad (3.1)$$

for all ξ, η, and where D^m denotes any derivative of order m.

Proof. We have

$$\begin{aligned}&|D^m\hat{f}(\xi)-D^m\hat{f}(\eta)|\\&\quad\leqslant\int_{\mathbf{R}^3}2\left|\sin\left(\frac{1}{2}|\xi-\eta|\cdot|v|\right)\right|\,|v|^m\,|f(v)|\,dv\\&\quad\leqslant2\int_{\mathbf{R}^3}|f(v)|\cdot|v|^m\phi(|v|)\,dv\,\sup_{|v|}\frac{|\sin(\frac{1}{2}|\xi-\eta|\,|v|)|}{\phi(|v|)}\end{aligned}$$

and the result follows immediately, since $|\sin(xy)/\phi(x)| \leqslant \max(xy, 1)/\phi(x) \leqslant 1/\phi(1/y)$. ∎

The reverse part is much more difficult to prove.

Lemma 3.2. Let ϕ and ψ be defined as in Lemma 3.1, and assume in addition that $\phi(r)$ is differentiable and that ϕ' (and ϕ) are of regular variation.[4] If f is nonnegative and the Fourier transform $\hat{f}$ satisfies the right-hand side estimate of (3.1), then

$$\int_{\mathbf{R}^3} f(v)\,|v|^m\,[\phi(|v|)]^{1/2}\,dv \leqslant C \int_{\mathbf{R}^3} f(v)\,|v|^m\,dv \tag{3.2}$$

where the constant C can be expressed in terms of ϕ.

Remark 3.1. The study of Abelian–Tauberian estimates is a very classical field of mathematics, and the literature is very extensive; a good source of results of this type is in ref. 4. Thus we dare not claim that these are new results, though in fact we have not been able to find precisely the estimate (3.2), and therefore a sketch of the proof will be given below.

Remark 3.2. A measurable function f is of regular variation if there is a real α, such that for all $x > 0$, $\lim_{t\to\infty} f(tx)/f(t) = x^\alpha$. In our case $\alpha = -1$ for ϕ'. Typical examples are $\phi(v) = \log(1+v)$ and $\phi(v) = \log\{1+\log[1+\log(1+v)]\}$.

Proof of Lemma 3.2. First we note that it is enough to consider the same problem in $\mathbf{R}^1$ and to assume that ϕ is smooth. Moreover, we consider here only the case $m = 0$. Write $g(v) = f(v) - \varphi(v)$, where $\varphi(v) = (2\pi)^{-1/2}\,\|f\|_{L^1}\exp(-v^2/2)$. Then $\hat{g}(0) = 0$, and $\hat{f}$ and $\hat{g}$ have the same continuity properties. Moreover, if $G(v) = \int_{-\infty}^{v} g(v')\,dv'$, then $\hat{G}(\xi) = \hat{g}(\xi)/\xi i$ for $\xi \neq 0$, and $G(v) \to 0$ as $|v| \to \infty$. Next let $\tilde{\phi}(v) = [\phi(v)]^{1/2}$ and let ϕ_n be a sequence of smooth, concave functions, coinciding with $\tilde{\phi}$ for small v, and $\phi_n(v) = n$ for suffciently large v. Then

$$\int_{-\infty}^{\infty} f(v)\,\tilde{\phi}(|v|)\,dv = \lim_{n\to\infty} \int_{-\infty}^{\infty} g(v)\,\phi_n(|v|)\,dv + \int_{-\infty}^{\infty} \varphi(v)\,\tilde{\phi}(|v|)\,dv \tag{3.3}$$

The second term is clearly bounded. To estimate the first term, we proceed by somewhat formal calculations, which can be made rigorous. Integrating partially, and then using Parseval's formula, we find that

$$\int_{-\infty}^{\infty} g(v)\,\phi_n(v)\,dv = -\int_{-\infty}^{\infty} G(v)\,\phi_n'(v)\,dv = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{i\xi}\,(\phi_n')^{\wedge}(\xi)\,d\xi \tag{3.4}$$

From the assumptions on ϕ, $(\phi'_n)\,\hat{}\,(\xi) \leqslant C/|\xi|$ for $|\xi| > 1$, and $|\hat{g}(\xi)| \leqslant 2\,\|f\|_{L^1}$. Therefore the contribution in (3.4) from $|\xi| > 1$ is bounded by $C\,\|f\|_{L^1}/\pi$, and the remaining part is bounded by

$$\left|\frac{1}{\pi}\int_{-1}^{1}\frac{\hat{g}(\xi)}{\xi}\int_0^\infty \phi'_n(v)\sin(v\xi)\,dv\,d\xi\right| \leqslant \frac{1}{\pi}\int_{-1}^{1}\frac{|\hat{g}(\xi)|}{|\xi|}\left|\int_0^\infty \phi'_n(v)\sin(v\xi)\,dv\right|d\xi \tag{3.5}$$

But for $\xi > 0$,

$$\left|\int_0^\infty \phi'_n(v)\sin(v\xi)\,dv\right| = \left|\frac{1}{\xi}\int_0^\infty \phi'_n\left(\frac{v}{\xi}\right)\sin(v)\,dv\right| \leqslant \frac{1}{\xi}\int_0^\pi \phi'_n\left(\frac{v}{\xi}\right)\sin(v)\,dv$$

which together with a change of variables $y = 1/\xi$ in (3.5), and the fact that ϕ_n is monotonously increasing with n, gives the estimate

$$\frac{1}{\pi}\int_1^\infty\left|\hat{g}\left(\frac{1}{y}\right)+\hat{g}\left(\frac{-1}{y}\right)\right|\int_0^\pi \tilde{\phi}'(yv)\sin(v)\,dv\,dy$$

We now invoke the estimate of $\hat{g}$ and the fact that the regular variation of $\tilde{\phi}'$ implies that[4]

$$\int_0^\pi \tilde{\phi}'(yv)\sin(v)\,dv \sim \tilde{\phi}'(y)\int_0^\pi \sin(v)/v\,dv \qquad \text{when} \quad y\to\infty$$

to obtain

$$\frac{1}{\pi}\int_1^\infty \frac{C}{\tilde{\phi}(v)^2}\int_0^\pi \tilde{\phi}'(yv)\sin(v)\,dv\,dy \leqslant \frac{C}{\pi}\int_1^\infty \frac{\tilde{\phi}'(v)}{\tilde{\phi}(v)^2}\,dv \leqslant \frac{C}{\tilde{\phi}(1)}$$

and the constant depends only on the specific choice of ϕ. ∎

After this we return to Eqs. (2.8) and (2.21) and hope to replace the factor $|\xi|^{2+\alpha}$ with $|\xi|^2\,\phi(\xi)$, where ϕ is some function, strictly increasing from 0 to ∞. The estimate (2.9) now becomes

$$\begin{aligned} A &= \int_{-\pi}^{\pi}\left(\left|\cos\frac{\theta}{2}\right|^2 \frac{\phi(|\xi|\cos(\theta/2))}{\phi(|\xi|)} + \left|\sin\frac{\theta}{2}\right|^2 \frac{\phi(|\xi|\sin(\theta/2))}{\phi(|\xi|)}\right)\sigma(\theta)\,d\theta \\ &< \int_{-\pi}^{\pi}\left[\cos\left(\frac{\theta}{2}\right)^2 + \sin\left(\frac{\theta}{2}\right)^2\right]\sigma(\theta)\,d\theta = 1 \end{aligned}$$

and again the strict inequality gives exponential decay by the Gronwall inequality. Equation (2.22) is modified in the same way. Also the estimates in Lemma 2.3 hold in this case if ϕ satisfies some weak conditions. There, in the proof we replace $|\cos(\theta/2)|^{2+\alpha} + |\sin(\theta/2)|^{2+\alpha}$ by

$$\left|\cos\frac{\theta}{2}\right|^2 \frac{\phi(|\xi|\cos(\theta/2))}{\phi(|\xi|)} + \left|\sin\frac{\theta}{2}\right|^2 \frac{\phi(|\xi|\sin(\theta/2))}{\phi(|\xi|)}$$

Then

$$\begin{aligned} A_k &= \frac{1}{\bar{\sigma}_k} \int_{-\pi}^{\pi} \left[1 - \sin^2\left(\frac{\theta}{2}\right)\left(\frac{\phi(|\xi|\cos(\theta/2))}{\phi(|\xi|)} - \frac{\phi(|\xi|\sin(\theta/2))}{\phi(|\xi|)}\right) \right. \\ &\quad \left. - 2\,|\xi| \sin^2\left(\frac{\theta}{4}\right) \frac{\phi'(|\xi|)}{\phi(|\xi|)} + \mathcal{O}\left(\sin^2\left(\frac{\theta}{4}\right)\right)\right] \min(\sigma(\theta), k)\, d\theta \\ &= 1 - \frac{\bar{A}_k}{\bar{\sigma}_k} \end{aligned}$$

and the sequence $\bar{A}_k$ converges just as in the previous case if $|\xi|\ \phi'(\xi)/\phi(\xi)$ is uniformly bounded (e.g., if ϕ is a slowly varying function). The same holds for the other estimates in Lemma 2.3.

The main result concerning uniform integrability at infinity of solutions to the Boltzmann equation follows immediately.

Theorem 3.3. Consider the Boltzmann equation (1.1), and assume that the initial data $f_0(v)$ satisfy the conditions of bounded mass and energy. For any $\varepsilon > 0$, there is an $R_\varepsilon < \infty$, such that

$$\int_{|v| > R_\varepsilon} f(v, t)\, |v|^2\, dv < \varepsilon$$

and this estimate holds uniformly for all $t \geqslant 0$.

Proof. According to the discussion in the beginning of the section, there is a function $\phi(r)$, strictly increasing to infinity, such that $\int_{\mathbf{R}^3} f_0(v)[1 + |v|^2\, \phi(|v|)]\, dv < \infty$. One possible construction of such a function which also satisfies the conditions of Lemma 3.2 is the following. Chose a sequence of numbers $x_i \to \infty$ such that

$$\int_{x_{i-1} \leqslant |v| < x_i} f(v)\, dv = 2^{-i}$$

(assume that the total mass of f is one), and then another sequence y_i such that for all i, $x_i \leqslant y_i$ and $y_i - y_{i-1} < y_{i+1} - y_i$. Let $\lambda_i = \log(y_{i+1}/y_i)$, and let

$$b_{i+1} = R^{-1}[\log\{1 + 1/(i+1)\}\ \lambda_i R[-b_i]/\{\log(1 + 1/(i)\}\ \lambda_{i+1})]$$

where $R[s] = (\exp(s) - 1)/s$. Then define a piecewise constant function $b(s) = b_i$ for $y_{i-1} \leqslant s < y_i$. Let $a(t) = \exp\{\int_1^t b(s)/s\, ds\}$, and finally $\phi(r) = \exp\{\int_1^r a(t)/t\, dt\}$. This construction guarantees that ϕ and ϕ' are of regular variation, and that ϕ is increasing, but sufficiently slowly. ∎

4. THE TANAKA METRIC AND THE KAC EQUATION

In this section, we introduce some concepts from probability theory that have direct connection with our problem of exponential convergence toward equilibrium.

Most of the definitions and proofs both of the present and of the next section hold when the underlying space is a separable metric space (S, d). Nevertheless, to avoid inessential difficulties, many proofs proofs will be restricted to $\mathbf{R}^d$, $d \geqslant 1$. The set of all probability measures on S will be denoted by $P(S)$. $C(S)$ denotes the Banach space of bounded continuous real-valued functions on S, with norm

$$\|\varphi\|_\infty = \sup\{|\varphi(x)|: x \in S\}$$

On $P(S)$ we put the usual weak-star topology TW^*, the weakest such that

$$P \to \int \varphi\, dP; \qquad P \in P(S)$$

is continuous for each $\varphi \in C(S)$.

It is known after Prokhorov[28, 37, 14] that TW^* on $P(S)$ is metrizable, and that this metrization can be done in different ways.

In our applications S is the space $\mathbf{R}^d$, $d \geqslant 1$. In this case, each probability measure P induces the distribution function F_P defined by

$$F_P(v) = P((-\infty, v_1), ..., (-\infty, v_d)), \qquad v = (v_1, ..., v_d) \in \mathbf{R}^d$$

and one of these metrics in $P(\mathbf{R}^d)$, $d \geqslant 1$, can be obtained by a convex functional introduced by Tanaka[34] in connection with the Kac equation. The functional T on P is defined by

$$T(F, G) = \inf E\{|\mathbf{X} - \mathbf{Y}|^2\}, \qquad F, G \in P \tag{4.1}$$

where the infimum is taken over all pairs of $\mathbf{R}^d$-valued random variables $\mathbf{X}$ and $\mathbf{Y}$ defined on a probability space (Ω, F, P) and distributed according to F and G, respectively.

An alternative definition of T is the following. Let $\Pi(F, G)$ be the set of all distribution functions L on $\mathbf{R}^{2d}$ having F and G as marginal distribution functions, where F and G are in P. Then

$$T(F, G) = \inf_{L \in \Pi} \int_{\mathbf{R}^{2d}} |v-w|^2 \, dL(v, w) \tag{4.2}$$

The analysis of Tanaka[34] was extended to the multidimensional case by Murata and Tanaka[27] and an important application to the Boltzmann equation for Maxwell molecules was finally presented in by Tanaka.[35] By means of classical properties of bivariate distributions with given marginals, first obtained by Hoeffding[22] and Fréchet,[17] simple proofs of the main properties of the Tanaka functional have recently been presented by Toscani[36] and Pulvirenti and Toscani.[29]

Let $x \wedge y = \min[x, y]$.

Theorem 4.1. Let $L^*(v, w) = F(v) \wedge G(w)$. Then, in $\Pi(F, G)$

$$L(v, w) \leqslant L^*(v, w) = F(v) \wedge G(w) \tag{4.3}$$

and

$$T(F, G) = \int_{\mathbf{R}^{2d}} |v-w|^2 \, dL^*(v, w) \tag{4.4}$$

Next, the existence of a pair of random variables with joint law L^* is proven by the following result.[36]

Theorem 4.2. For any $F, G \in P$ there exist measurable functions $\mathbf{A}_F, \mathbf{A}_G\colon [0, 1] \to \mathbf{R}^d$ such that $(\mathbf{A}_F(U), \mathbf{A}_G(U))$, where U is a uniform random variable, has joint distribution function L^*.

Owing to (4.3) and by Theorem 4.1, the main properties of the functional T are derived. In particular the following holds.[35]

Theorem 4.3. Let

$$v(F, G) = [T(F, G)]^{1/2}; \qquad F, G \in P(\mathbf{R}^d)$$

Then v is a metric and metrizes TW^* on $P(\mathbf{R}^d)$.

Let $f(v)$ and $g(v)$, $v \in \mathbf{R}$, be probability densities with finite moments of order $2+\alpha$, for some $\alpha > 0$, and suppose

$$\int_{\mathbf{R}} |v|^{2+\alpha} f(v) \, dv \leqslant M; \qquad \int_{\mathbf{R}} |v|^{2+\alpha} g(v) \, dv \leqslant M \tag{4.5}$$

Let $F(v)$ and $G(v)$ be the induced probability distributions. A direct consequence of Theorem 4.1 in the one-dimensional case is that, denoting by F^{-1} and G^{-1} the inverse functions of F and G, respectively,

$$T(F, G) = \int_{\mathbf{R}^2} |v-w|^2 \, dL^*(v, w) = \int_0^1 |F^{-1}(t) - G^{-1}(t)|^2 \, dt \tag{4.6}$$

For any $R>0$, (4.5) implies

$$\int_{(|v|>R) \cup (|w|>R)} |v-w|^2 \, dL^*(v, w) \leqslant 2M/R^{\alpha} \tag{4.7}$$

Moreover,

$$\begin{aligned}
&\int_{\mathbf{R}^2} |v-w|^2 \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, dL^*(v, w) \\
&\quad \leqslant 2R \int_{\mathbf{R}^2} |v-w| \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, dL^*(v, w) \\
&\quad = 2R \int_0^1 |F^{-1}(t) - G^{-1}(t)| \, I_{\{|F^{-1}(t)| \leqslant R\} \cap \{|G^{-1}(t)| \leqslant R\}} \, dt
\end{aligned} \tag{4.8}$$

The value of the integral in (4.8) is given by the measure of the area between the two distribution functions F and G included in the strip $[-R, R]$. In fact

$$\begin{aligned}
&\int_0^1 |F^{-1}(t) - G^{-1}(t)| \, I_{\{|F^{-1}(t)| \leqslant R\} \cap \{|G^{-1}(t)| \leqslant R\}} \, dt \\
&\quad = \int_{-R}^{R} |F(v) - G(v)| \, dv
\end{aligned} \tag{4.9}$$

On the other hand, for any $\beta > 0$

$$\begin{aligned}
&\int_{-R}^{R} |F(v) - G(v)| \, dv \\
&\quad = \int_{-R}^{R} |F(v) - G(v)| \, I_{|F(v) - G(v)| \leqslant \beta} \, dv \\
&\qquad + \int_{-R}^{R} |F(v) - G(v)| \, I_{|F(v) - G(v)| > \beta} \, dv \\
&\quad \leqslant 2R\beta + \frac{1}{\beta} \int_{\mathbf{R}} |F(v) - G(v)|^2 \, dv
\end{aligned}$$

By Parseval's formula,

$$\int_{\mathbf{R}} |F(v)-G(v)|^2\,dv = \frac{1}{2\pi}\int_{\mathbf{R}} |\hat{F}(\xi)-\hat{G}(\xi)|^2\,d\xi$$

But

$$\hat{F}(\xi)-\hat{G}(\xi) = \frac{\hat{f}(\xi)-\hat{g}(\xi)}{i\xi}$$

and therefore

$$\begin{aligned}\int_{\mathbf{R}} &|F(v)-G(v)|^2\,dv\\ &= \frac{1}{2\pi}\int_{\mathbf{R}} \frac{|\hat{f}(\xi)-\hat{g}(\xi)|^2}{\xi^2}\,d\xi\\ &\leqslant \frac{1}{2\pi}\left(\int_{-N}^{N} \frac{|\hat{f}(\xi)-\hat{g}(\xi)|^2}{\xi^2}\,d\xi + 2\int_{|\xi|\geqslant N}\frac{1}{\xi^2}\,d\xi\right)\end{aligned} \tag{4.10}$$

Finally, we get the estimate

$$T(F,G) \leqslant 2\,\frac{M}{R^\alpha} + 4R^2\beta + \frac{2R}{\beta}\left(\frac{N}{\pi}\sup_{|\xi|\leqslant N}\frac{|\hat{f}(\xi)-\hat{g}(\xi)|^2}{\xi^2} + \frac{1}{\pi N}\right) \tag{4.11}$$

Let us fix $\varepsilon > 0$, and take

$$R = \left(\frac{8M}{\varepsilon}\right)^{1/\alpha}; \qquad \beta = \frac{\varepsilon}{16}\left(\frac{\varepsilon}{8M}\right)^{2/\alpha}; \qquad N = \frac{128}{\pi\varepsilon}\left(\frac{8M}{\varepsilon}\right)^{3/\alpha} \tag{4.12}$$

Then, provided

$$\sup_{|\xi|\leqslant N}\frac{|\hat{f}(\xi)-\hat{g}(\xi)|^2}{\xi^2} \leqslant \frac{\pi^2\varepsilon^2}{8\cdot 16^2}\left(\frac{\varepsilon}{8M}\right)^{6/\alpha} \tag{4.13}$$

it follows from (4.11) that

$$T(F,G) \leqslant \varepsilon \tag{4.14}$$

Now, consider that, for any $\gamma > 0$

$$\sup_{|\xi|\leqslant N}\frac{|\hat{f}(\xi)-\hat{g}(\xi)|^2}{\xi^2} \leqslant N^\gamma \left\|\frac{\hat{f}(\xi)-\hat{g}(\xi)}{\xi^{2+\gamma}}\right\|_\infty \tag{4.15}$$

Thus, inequality (4.13) is verified if

$$\left[\frac{128}{\pi\varepsilon}\left(\frac{8M}{\varepsilon}\right)^{3/\alpha}\right]^{\gamma} N^{\gamma} \left\|\frac{\hat{f}(\xi)-\hat{g}(\xi)}{\xi^{2+\gamma}}\right\|_{\infty} = \frac{\pi^2\varepsilon^2}{8\cdot 16^2}\left(\frac{\varepsilon}{8M}\right)^{6/\alpha} \tag{4.16}$$

Put

$$\begin{aligned} d &= d(\alpha, \gamma) = 3+\gamma+\frac{6+3\gamma}{\alpha} \\ C &= C(M, \alpha, \gamma) = 16\cdot 128^{1+\gamma}(8M)^{(6+3\gamma)/\alpha} \end{aligned} \tag{4.17}$$

Then, by (4.16),

$$\varepsilon = C^{1/d}\left\|\frac{\hat{f}(\xi)-\hat{g}(\xi)}{\xi^{2+\gamma}}\right\|_{\infty}^{1/d} \tag{4.18}$$

substitution of (4.18) into (4.15) proves the following result.

Theorem 4.4. Let f, g be probability densities that satisfy (4.5) for some $\alpha > 0$. Then for any $\gamma > 0$, there are C and $d > 0$ such that

$$T(F, G) \leqslant C^{1/d}\left\|\frac{\hat{f}(\xi)-\hat{g}(\xi)}{\xi^{2+\gamma}}\right\|_{\infty}^{1/d} \tag{4.19}$$

The constants C and d are given by (4.17), and so depend only on α, γ, and M.

Coupling the result of Theorem 4.4 with Lemma 2.1 gives the following result.

Theorem 4.5. Let $f(v, t)$ be the solution to the initial value problem for the Kac equation with or without cutoff, with initial density $f_0(v)$ such that, for some $\gamma > 0$,

$$\int_{\mathbf{R}} |v|^{2+\gamma} f_0(v)\, dv = M_0 < \infty \tag{4.20}$$

Then, $f(v, t)$ is exponentially convergent toward equilibrium in Tanaka metric, and

$$v(f(t), \omega) \leqslant C^{1/2d}\left\|\frac{\hat{f}_0(\xi)-\hat{\omega}(\xi)}{\xi^{2+\gamma}}\right\|_{\infty}^{1/2d} \exp\left(-\frac{c_\gamma}{2d}t\right) \tag{4.21}$$

Proof. Since the initial density satisfies (4.20), there is M_γ such that

$$\int_{\mathbf{R}} |v|^{2+\gamma} f(v, t)\, dv \leqslant M_\gamma; \qquad \int_{\mathbf{R}} |v|^{2+\gamma} \omega(v)\, dv \leqslant M_\gamma$$

By Lemma 2.1 from Section 2, we have

$$\left\| \frac{\hat{f}(\xi, t) - \hat{\omega}(\xi)}{\xi^{2+\gamma}} \right\|_\infty \leqslant \left\| \frac{\hat{f}_0(\xi) - \hat{\omega}(\xi)}{\xi^{2+\gamma}} \right\|_\infty \exp^{-c_\gamma t}$$

and the result follows by Theorem 4.5. ▌

5. OTHER METRICS AND EXPONENTIAL CONVERGENCE OF A MAXWELL GAS

In Section 4, we obtained exponential convergence to equilibrium for the Kac equation in Tanaka metric. There, in consequence of formula (4.6) applied to the one-dimensional case, an explicit rate of convergence was found. We remark that, as emphasized by Carlen and Carvalho,[8] the evaluation of the exponential rate of convergence toward equilibrium is of great importance in passing to the fluid-dynamical level.

In higher dimensions of the velocity variable, Theorem 4.2 is not useful for obtaining an explicit expression for T. Therefore we try a different approach. The starting point will be the introduction of other metrics on $P(S)$.

For any $x \in S$ and $U \subset S$, let

$$d(x, U) = \inf\{d(x, y) : y \in U\} \tag{5.1}$$

and for $\delta \geqslant 0$ let

$$U^\delta = \{x \in S : d(x, U) < \delta\}; \qquad U^{\delta]} = \{x \in S : d(x, U) \leqslant \delta\} \tag{5.2}$$

Given P and Q in $P(S)$, let

$$\sigma(P, Q) = \inf\{\varepsilon > 0 : P(A) \leqslant Q(A^\varepsilon) + \varepsilon \text{ for all closed } A \subset S\} \tag{5.3}$$

where $P(A) = \int_S I_A \, dP$, and

$$\rho(P, Q) = \max\{\sigma(P, Q), \sigma(Q, P)\} \tag{5.4}$$

Then, ρ is a metric and metrizes TW^* on $P(S)$.

This metric is known as Prokhorov metric, and was introduced in ref. 28 for S complete. A subsequent generalization to a separable metric space was obtained by Dudley.[15] For S complete, Strassen[33] proved the striking and important result that, if $P, Q \in P(S)$, the Prokhorov distance $\rho(P, Q)$ is exactly the minimum distance "in probability" between random variables distributed according to P and Q.

Let us remark that we may replace A^{ε} by $A^{\varepsilon]}$ in the definition of σ without changing its value. Also, we may replace "all closed A" by "all Borel sets B," since, if A is the closure of B, $A^{\varepsilon} = B^{\varepsilon}$ and $A^{\varepsilon]} = B^{\varepsilon]}$.

Strassen[33] proved that, if $P, Q \in P(S)$ and $\alpha, \beta > 0$, then $P(A) \leqslant Q(A^{\alpha}) + \beta$ for all closed A if and only if the same condition hold with P and Q interchanged. Thus $\sigma(P, Q) = \sigma(Q, P) = \rho(P, Q)$.

Let $\Pi(P, Q)$ be the set of all probability measures on $S \times S$ having P and Q as marginal measures. Then, the following holds.[15]

Theorem 5.1. Let S be a separable metric space, $P, Q \in P(S)$, $\alpha \geqslant 0$, and $\beta \geqslant 0$. Then, the following are equivalent:

(a) $P(A) \leqslant Q(A^{\alpha]}) + \beta$ for all closed $A \subset S$.

(b) For any $\varepsilon > 0$ there is $L \in \Pi(P, Q)$ such that $L(d(x, y) > \alpha + \varepsilon) \leqslant \beta + \varepsilon$.

Theorem 5.1 is all one needs to get a relation between Prokhorov and Tanaka metrics in $P(\mathbf{R}^d)$, $d \geqslant 1$. As in the previous section, given a probability density f, we will denote by the capital letter F the corresponding distribution function. We have the following result.

Theorem 5.2. Let $F, G \in P(\mathbf{R}^d)$, $d \geqslant 1$, and suppose that f, g satisfy (4.5) for some $\alpha > 0$. Then

$$T(F, G) \leqslant (2M + 8)\, \rho(F, G)^{\alpha/(\alpha+2)} + 4\rho(F, G)^2 \tag{5.5}$$

Proof. By Theorem 4.1, and thanks to (4.5), for any $R > 0$ and $L \in \Pi(F, G)$,

$$\begin{aligned} T(F, G) &= \int_{\mathbf{R}^{2d}} |v - w|^2 \, dL^*(v, w) \\ &\leqslant \int_{\mathbf{R}^{2d}} |v - w|^2 \, dL(v, w) \\ &\leqslant \frac{2M}{R^{\alpha}} + \int_{\mathbf{R}^{2d}} |v - w|^2 \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, dL(v, w) \end{aligned}$$

Assume that $\rho(F, G) = \delta$. Then, according to Theorem 5.1, we can choose $L \in \Pi(F, G)$ such that $L(|v - w| > 2\delta) \leqslant 2\delta$. For such L

$$\begin{aligned} &\int_{\mathbf{R}^{2d}} |v - w|^2 \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, dL(v, w) \\ &\quad = \int_{\mathbf{R}^{2d}} |v - w|^2 \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, I_{\{|v - w| > 2\delta\}} \, dL(v, w) \\ &\qquad + \int_{\mathbf{R}^{2d}} |v - w|^2 \, I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} \, I_{\{|v - w| \leqslant 2\delta\}} \, dL(v, w) \end{aligned}$$

$$\leqslant 4\delta^2 \int_{\mathbf{R}^{2d}} I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} I_{\{|v-w| > 2\delta\}} \, dL(v, w)$$

$$+ 4R^2 \int_{\mathbf{R}^{2d}} I_{\{|v| \leqslant R\} \cap \{|w| \leqslant R\}} I_{\{|v-w| > 2\delta\}} \, dL(v, w) \leqslant 4\delta^2 + 8R^2\delta$$

The result follows by choosing $R = \delta^{-1/(\alpha+2)}$. ∎

We shall now use Theorem 5.1 to compare the Prokhorov metric with other metrizations of $P(\mathbf{R}^d)$. These metrics are obtained by generalizing another metrization of $P(S)$ given in ref. 15. Given $n \geqslant 1$, let

$$C^n = C^n(\mathbf{R}^d) = \left\{\phi \colon \mathbf{R}^d \to \mathbf{R} \colon \sup_{v \in \mathbf{R}^d} \sum_{k=1}^{n} |D^k f(v)| < \infty\right\}$$

endowed with the sup-norm, which we denote by $\|\cdot\|_n$. Given $R > 0$, let $B_R = \{v \in \mathbf{R}^d \colon |v| < R\}$, and let $C_R^n = C^n(B_R)$.

Given $F \in P(\mathbf{R}^d)$, we define

$$\|F\|_n^* = \sup\left\{\left|\int_{\mathbf{R}^d} \phi \, dF\right|; \phi \in C^n, \|\phi\|_n \leqslant 1\right\} \tag{5.6}$$

An analogous definition can be made when $\mathbf{R}^d$ is substituted by B_R. In this case the norm will be indicated by $\|\cdot\|_{n,R}$. The metric $\|F-G\|_n^*$ metrizes TW^* on $P(\mathbf{R}^d)$. This will be clear after comparison with the Prokhorov metric. We have the following result.

Lemma 5.3. Let $F, G \in P(\mathbf{R}^d)$, $\alpha \geqslant 0$, and $\beta \geqslant 0$. Then, if (a) of Theorem 5.1 holds,

$$\|F - G\|_n^* \leqslant 2 \max\{\alpha, \beta\} \tag{5.7}$$

Proof. Because of the equivalence between (a) and (b) of Theorem 5.1, we can find $L \in \Pi(F, G)$ such that $L(|v-w| > \alpha + \varepsilon) \leqslant \beta + \varepsilon$. Take a pair of random variables $(\mathbf{X}, \mathbf{Y})$, distributed according to F and G, respectively, with joint distribution L. Then, for any $\phi \in C^n$,

$$\left|\int_{\mathbf{R}^d} \phi d(F-G)\right|$$

$$= |E(\phi(\mathbf{X})) - E(\phi(\mathbf{Y}))|$$

$$= \left|\int_{\mathbf{R}^{2d}} [\phi(v) - \phi(w)] \, dL(v, w)\right|$$

$$= \left| \int_{\mathbf{R}^{2d}} [\phi(v) - \phi(w)]\, I_{\{|v-w| > \alpha+\varepsilon\}}\, dL(v, w) + \int_{\mathbf{R}^{2d}} [\phi(v) - \phi(w)]\, I_{\{|v-w| \leqslant \alpha+\varepsilon\}}\, dL(v, w) \right|$$

$$\leqslant \int_{\mathbf{R}^{2d}} |\phi(v) - \phi(w)|\, I_{\{|v-w| > \alpha+\varepsilon\}}\, dL(v, w) + \int_{\mathbf{R}^{2d}} \left\{ \sup_{v \neq w} \frac{|\phi(v) - \phi(w)|}{|v-w|} \right\} |v-w|\, I_{\{|v-w| \leqslant \alpha+\varepsilon\}}\, dL(v, w)$$

$$\leqslant 2\, \|\phi\|_\infty \int_{\mathbf{R}^{2d}} I_{\{|v-w| > \alpha+\varepsilon\}}\, dL(v, w) + (\alpha+\varepsilon) \sup_{v \neq w} \frac{|\phi(v) - \phi(w)|}{|v-w|}$$

$$\leqslant 2(\beta+\varepsilon)\, \|\phi\|_\infty + (\alpha+\varepsilon) \sup_{v \neq w} \frac{|\phi(v) - \phi(w)|}{|v-w|}$$

Letting ε to 0, we get the desired conclusion. ∎

Lemma 5.4. Let $F, G \in P(\mathbf{R}^d)$, A be a compact set of $\mathbf{R}^d$, $\alpha \geqslant 0$, and $\beta \geqslant 0$, and

$$F(A) > G(A^\beta) + \alpha$$

Then

$$\|F - G\|_n^* \geqslant \frac{2\alpha\beta^n}{c_n + \beta^n} \tag{5.8}$$

where c_n is a positive constant depending on n and d only.

Proof. According to Hörmander (ref. 23, Theorem 1.4.1, p. 25), given a compact set $A \subset \mathbf{R}^d$ and a constant $\beta > 0$, there is $\psi \in C_0^\infty(A^\beta)$ with $0 \leqslant \psi \leqslant 1$, $\psi(v) = 1$ for all $v \in A$, and $|D^k \psi| \leqslant c_k \beta^{-k}$, where c_k depends on $k \geqslant 1$, n, and d, but not on the compact set A. Hence, we can choose $\phi \in C^n$ with $\|\phi\|_n \leqslant 1 + c_n \beta^{-n}$, where c_n is a new constant, and $\phi(v) = 1$ if $v \in A$, $\phi(v) = -1$ if $v \in \mathbf{R}^d \backslash A^\beta$. Then

$$\left(1 + \frac{c_n}{\beta^n}\right) \|F - G\|_n^* \geqslant \int_{\mathbf{R}^d} \phi\, d(F-G) = \int_{\mathbf{R}^d} (\phi + 1)\, d(F-G) = \int_{A^\beta} (\phi+1)\, d(F-G) \geqslant 2[F(A) - G(A^\beta)] \geqslant 2\alpha \quad ∎$$

Taking $\alpha = \beta = \delta$ in (5.8) gives

$$\|F-G\|_n^* \geqslant \frac{2\delta^{n+1}}{c_n + \delta^n} \tag{5.9}$$

Suppose $\delta < c_n^{1/n}$. Then by (5.9)

$$\|F-G\|_n^* \geqslant \frac{\delta^{n+1}}{c_n}$$

In the opposite case,

$$\|F-G\|_n^* \geqslant \delta$$

Hence we have the following result.

Corollary 5.5. Let $F, G \in P(\mathbf{R}^d)$. Then, for each $n > 0$ there exists a constant c_n, depending only on n and d, such that

$$\rho(F, G) \leqslant \max\{c_n[\,\|F-G\|_n^*]^{1/(n+1)};\ \|F-G\|_n^*\} \tag{5.10}$$

Lemma 5.3 and Corollary 5.5 together imply that $\|\cdot\|_n^*$ and ρ define the same weak-star uniformity on $P(\mathbf{R}^d)$. We remark that the same conclusion can be drawn for the spaces B_R. In fact, we only have to substitute B_R to $\mathbf{R}^d$ in the proofs of the previous lemmas.

Lemma 5.6. Let $F, G \in P(\mathbf{R}^d)$, such that, for $R > 0$ and $\gamma > 0$,

$$\int_{|v|>R} dF(v) \leqslant M/R^\gamma; \qquad \int_{|v|>R} dG(v) \leqslant M/R^\gamma \tag{5.11}$$

and for $\gamma > 0$

$$\begin{aligned}\rho_R &= \inf\{\varepsilon > 0\colon F(A) \leqslant G(A^\varepsilon) + \varepsilon \text{ for all closed } A \subset B_R\} \\ &\leqslant dR^\beta e^{-Kt}\end{aligned} \tag{5.12}$$

where $\beta > 0$ and $k > 0$. Then

$$\rho(F, G) \leqslant (d+M) \exp\left(-\frac{\gamma}{\beta+\gamma} Kt\right) \tag{5.13}$$

Proof. For any closed set $A \subset \mathbf{R}^d$, thanks to (5.12) we obtain

$$\begin{aligned}F(A) &\leqslant F(A \cap B_R) + M/R^\gamma \\ &\leqslant G([A \cap B_R]^{\rho_R}) + \rho_R + M/R^\gamma \\ &\leqslant G([A \cap B_R]^{\rho_R + M/R^\gamma}) + \rho_R + M/R^\gamma\end{aligned}$$

Hence, by definition (5.3),

$$\rho(F, G) \leqslant dR^{\beta} e^{-Kt} + M/R^{\gamma}$$

Take $R = \exp\{(Kt)/(\beta+\gamma)\}$, and (5.13) follows. ▮

Lemma 5.7. Let $R > 0$. Then, given $\alpha \geqslant -2$, for $n > d + \alpha + 3$

$$\|F - G\|_{n,R}^{*} \leqslant D(R+1)^{d+1} \left\| \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^{2+\alpha}} \right\|_{\infty} \tag{5.14}$$

where

$$D = D(n, d, \alpha) = 2 \int_{\mathbf{R}^d} \frac{dv}{(1+|v|)^{d+1}} \int_{\mathbf{R}^d} \frac{|v|^{\alpha+2}\, dv}{1+|v|^n} \tag{5.15}$$

Proof. Given $R > 0$, let us take a function $\phi \in C_R^n$ such that $\operatorname{supp} \phi \in B_R$. Given the probability densities f, g, by Parseval's formula, we have

$$\int_{\mathbf{R}^d} \phi(v)[f(v) - g(v)]\, dv = \int_{\mathbf{R}^d} \hat{\phi}(\xi)[\hat{f}(\xi) - \hat{g}(\xi)]\, d\xi$$

Moreover, since $\operatorname{supp} \phi \in B_R$, by the classical inequality

$$\sup_{\xi}\{(1+|\xi|^k)\, \hat{\phi}(\xi)\} \leqslant c_d \sup_{v}\{(1+|v|)^{d+1}\, |D^k \phi(c)|\}$$

$$c_d = \int_{\mathbf{R}^d} \frac{dv}{(1+|v|)^{d+1}}$$

we obtain, for $k \leqslant n$,

$$\sup_{\xi} |\xi^k \hat{\phi}(\xi)| \leqslant c_d (1+R)^{d+1}\, \|\phi\|_n \tag{5.16}$$

By the previous inequality

$$\begin{aligned}
&\int_{\mathbf{R}^d} \hat{\phi}(\xi)[\hat{f}(\xi) - \hat{g}(\xi)]\, d\xi \\
&\quad = \int_{\mathbf{R}^d} |\hat{\phi}(\xi)| \cdot |\xi|^{\alpha+2} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^{\alpha+2}}\, d\xi \\
&\quad \leqslant \int_{\mathbf{R}^d} |\hat{\phi}(\xi)| \cdot |\xi|^{\alpha+2}\, d\xi \left\| \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^{2+\alpha}} \right\|_{\infty} \\
&\quad \leqslant \left\| \frac{\hat{f}(\xi) - \hat{g}(\xi)}{\xi^{2+\alpha}} \right\|_{\infty} \sup_{\xi}\{|\hat{\phi}(\xi)|\, (1+|\xi|^n)\} \int_{\mathbf{R}^d} \frac{|v|^{\alpha+2}\, dv}{1+|v|^n}
\end{aligned}$$

Thus, choosing $n > d + \alpha + 3$, by definition (5.6) the result follows. ▮

Let us denote by $\rho_R(F, G)$ the Prokhorov distance when F, G are restricted to B_R, and suppose

$$c_n[\|F-G\|^*_{n,R}]^{1/(n+1)} \geqslant \|F-G\|^*_{n,R}$$

Then, by Corollary 5.5 and Lemma 5.7

$$\rho_R(F, G) \leqslant c_n D^{1/(n+1)}(R+1)^{(d+1)/(n+1)} \left\| \frac{\hat{f}(\xi)-\hat{g}(\xi)}{\xi^{2+\alpha}} \right\|_\infty^{1/(n+1)} \tag{5.17}$$

The next step is to let $F=F(t)$ be the distribution corresponding to f, a solution of the Boltzmann equation, and $G=\Omega$, the equilibrium distribution. Because of the pressure term in Lemma 2.2, the estimate for $\hat{f}-\hat{\omega}$ does not quite fit into (5.17), but since $\hat{P}$ is bounded and supported in $\{|\xi| \leqslant 1\}$, we can simply replace $\|(\hat{f}-\hat{\omega})/|\cdot|^{2+\alpha}\|_\infty$ by $\|\Phi_1/|\cdot|^{2+\alpha}\|_\infty + \|\hat{P}\|_\infty$. Denote $\hat{P}(\cdot, 0)=\hat{P}_0$ and $\Phi_1(\cdot, 0)=\Phi_{1,0}$. Then Lemma 2.2 gives

$$\begin{aligned} &\rho_R(F(t), \Omega) \\ &\quad \leqslant c_n D^{1/(n+1)}(R+1)^{(d+1)/(n+1)} \left(C_1 + \left\| \frac{\Phi_{1,0}}{|\cdot|^{2+\alpha}} \right\|_\infty + \|\hat{P}_0\|_\infty \right)^{1/(n+1)} \\ &\qquad \times \exp\left(\frac{c_\alpha}{n+1}\right) \end{aligned}$$

Take $\gamma = 2+\alpha$, $\beta=(d+1)/(n+1)$, and $K=c_\alpha/(n+1)$. By Lemma 5.6 we obtain

$$\begin{aligned} &\rho_R(R(t), \Omega) \\ &\quad \leqslant c_n D^{1/(n+1)} \left[\left(C_1 + \left\| \frac{\Phi_{1,0}}{|\cdot|^{2+\alpha}} \right\|_\infty + \|\hat{P}_0\|_\infty \right)^{1/(n+1)} + M \right] \\ &\qquad \times \exp\left\{ -\frac{(2+\alpha)\, c_\alpha}{d+1+(2+\alpha)(n+1)}\, t \right\} \end{aligned} \tag{5.18}$$

If instead

$$c_n[\|F-G\|^*_{n,R}]^{1/(n+1)} \leqslant \|F-G\|^*_{n,R}$$

we obtain

$$\begin{aligned} \rho_R(F(t), \Omega) \leqslant D &\left[\left(C_1 + \left\| \frac{\Phi_{1,0}}{|\cdot|^{2+\alpha}} \right\|_\infty + \|\hat{P}_0\|_\infty \right) + M \right] \\ &\times \exp\left\{ -\frac{(2+\alpha)\, c_\alpha}{d+1+2+\alpha}\, t \right\} \end{aligned} \tag{5.19}$$

Finally we have the following result.

Theorem 5.8. Let $f(v, t)$ be the solution to the initial value problem for the Boltzmann equation for Maxwell molecules with initial density $f_0(v)$ such that, for some $\alpha > 0$,

$$\int_{\mathbf{R}} |v|^{2+\alpha} f_0(v)\, dv = M_0 < \infty \tag{5.20}$$

Then, $f(v, t)$ is exponentially convergent toward equilibrium in Prokhorov metric, and for $n > d + \alpha + 3$ the following bound holds:

$$\begin{aligned} &\rho_R(F(t), \Omega) \\ &\quad \leqslant \max\left\{ c_n D^{1/(n+1)} \left[\left(C_1 + \left\| \frac{\Phi_{1,0}}{|\cdot|^{2+\alpha}} \right\|_\infty + \|\hat{P}_0\|_\infty \right)^{1/(n+1)} + M \right];\right. \\ &\qquad\qquad \left. D\left[\left(C_1 + \left\| \frac{\Phi_{1,0}}{|\cdot|^{2+\alpha}} \right\|_\infty + \|\hat{P}_0\|_\infty \right) + M \right] \right\} \\ &\qquad \times \exp\left\{ -\frac{(2+\alpha)\, c_\alpha}{d+1+(2+\alpha)(n+1)}\, t \right\} \end{aligned} \tag{5.21}$$

where

$$\hat{P}_0(\xi) = \sum_{i,j} \left(\frac{\partial^2}{\partial_i \partial_j} \hat{f}_0(\xi) - \delta_{i,j} \frac{1}{3} \nabla^2 \hat{f}_0(\xi) \right) 1_{\{|\xi| \leqslant 1\}}$$

and

$$\Phi_{1,0} = \hat{f}_0(\xi) - \omega(\xi) - \hat{P}_0$$

Remark 5.1. The relation between the Tanaka distance and the Prokhorov distance given in Theorem 5.2 implies that the convergence to equilibrium is exponential also when measured in the Tanaka metric. This holds in the case when the initial data possess $2 + \alpha$ moments; an explicit formula corresponding to (5.5) can be found also in the more general case treated in Section 3, and thus it is also possible to estimate the rate of trend to equilibrium in that case. However, one does not necessarily obtain an exponential estimate.

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